[2]Wu, J.H.T., Neemeh, R. A., and Ostrowski, P. P., "Experimental Studies of the Production of Converging Cylindrical Shock Waves," *AIAA Journal*, Vol. 18, Jan. 1980, pp. 47-48.

[3]Perry, R. W. and Kantarowitz, A., "The Production and Stability of Converging Shock Waves," *Journal of Applied Physics*, Vol. 22, July 1951, pp. 878-886.

[4]Ben-Dor, G. and Glass, I. I., "Domains and Boundaries of Non-stationary Oblique Shock-Wave Reflexions. 1. Diatomic Gas," *Journal of Fluid Mechanics*, Vol. 92, Pt. 3, June 1979, pp. 459-496.

[5]Ben-Dor, G. and Glass, I. I., "Domains and Boundaries of Non-stationary Oblique Shock-Wave Reflexions. 2. Monatomic Gas," *Journal of Fluid Mechanics*, Vol. 96, Pt. 4, Feb. 1980, pp. 735-756.

[6]Ben-Dor, G., "Regions and Transitions of Non-stationary Oblique Shock Wave Diffractions in Perfect and Imperfect Gases," UTIAS Rept. No. 232, Aug. 1978.

[7]Ben-Dor, G., "Analytical Solution of Double Mach Reflection," *AIAA Journal*, Vol. 18, Sept. 1980, pp. 1036-1043.

AIAA 81-4094

Dimensional Analysis of Electrical Conductivity of Nonideal Classical and Quantum Plasmas

H. E. Wilhelm*
University of Florida, Gainesville, Fla.

Introduction

IN recent years, measurements of the electrical conductivity of nonideal plasmas have been reported.[1-10] In spite of the availability of an approximate kinetic equation for nonideal plasmas,[11] which considers spatial and temporal correlations in the collision operator, satisfactory theoretical explanations of the experimental conductivity data on nonideal plasmas are missing to date. The degree of the nonideality of a fully ionized plasma is measured in terms of the (dimensionless) interaction parameter γ, which represents the ratio of average Coulomb interaction ($Ze^2n^{1/3}$) and thermal (KT) energies (n = electron density, Z = ion charge number, e = elementary charge),

$$\gamma = Ze^2n^{1/3}/KT$$

The conductivity theory of ideal, fully ionized plasmas[12] agrees with the experimental data only if $\gamma \ll 1$. For moderately nonideal, $0.1 < \gamma \lesssim 1$, plasmas, the ideal conductivity theory yields much too large conductivity values. The ideal conductivity theory breaks down at higher electron densities because the Debye radius $D = [Z/4\pi(1+Z)]^{1/2} \gamma^{-1/2} n^{-1/3}$ loses its physical meaning as an electric shielding length and upper impact parameter when the number of electrons in the Debye sphere is no longer large compared with one. For typical nonideal conditions, $n > 10^{20}\text{cm}^{-3}$ and $T = 10^4$ K, the Debye radius is $D < 10^{-8}$cm (i.e., is smaller than the atomic diameter) which shows that the electric shielding concept is not applicable to proper nonideal plasmas, $\gamma \gtrsim 1$. Another reason for the inapplicability of the ideal conductivity theory to proper nonideal plasmas is its assumption of successive, small binary interactions, whereas in reality a conduction electron experiences many-body interactions for $\gamma \gtrsim 1$.

Received May 8, 1980; revision received Sept. 2, 1980.

*Professor of Engineering Sciences, Department of Engineering Sciences.

We apply dimensional theory to the derivation of new formulas for the electrical conductivity of (nonrelativistic) ideal and nonideal, classical and quantum plasmas. In the most general case of an electron plasma, the electrical conductivity σ is a function of the characteristic plasma parameters of the form ($\hbar$ is Planck's constant and m is the electron mass)

$$\sigma = C_\sigma e^p m^q \hbar^r (KT)^s n^t$$

The coefficient C_σ is in general a dimensionless function of the plasma parameters, which is not a simple power law, i.e., C_σ cannot be determined by dimensional analysis. C_σ is a true constant of order 10^0 if many particle interactions are dominant, whereas C_σ is a slowly varying function of order 10^{-1} if binary interactions are dominant.[12] Since the transition from binary to many particle interactions is continuous and smooth, it is unlikely that C_σ could behave as a strongly varying function in intermediate situations. Thus, dimensional theory gives the more interesting, strong power law dependence of the conductivity on the relevant plasma parameters. As special cases of the general conductivity formula, the known conductivity formulas for ideal and metallic plasmas are obtained.

Theoretical Foundations

In a system of reference in which magnetic fields are absent, a linear electric current response $j = \sigma E$ exists, provided that the generating electric field E is sufficiently weak. For any gaseous, liquid, or solid medium, the electrical conductivity $\sigma = |j|/|E|$ is given by

$$\sigma = (ne^2/m)\tau \tag{1}$$

where τ is the (average) momentum relaxation time of the current carriers. Equation (1) holds for any perturbed Maxwell or Fermi distribution of the electrons.

Dimensional analysis is based on the axioms of Dupre.[13] By axiom 1, a general relation may exist between two quantities a and b only when the two quantities have the same dimension. By axiom 2, the ratio of the magnitude of two like quantities a and b is independent of the units used in their measurement, provided that the same units are used for evaluating each.

In general, any measurable quantity σ (the secondary quantity) can be expressed in terms of those appropriate quantities a_i, $i = 1,2...M$ (the primary quantities), which affect the magnitude of σ. The general relationship between the magnitude of the secondary quantity σ and the magnitudes of the primary quantities a_i is a function of the M arguments of the form

$$\sigma = f(a_1, a_2, a_3, \ldots a_M) \tag{2}$$

Application of the axioms 1 and 2 to Eq. (2) demonstrates that the functional relation $f(a_i)$ is of the form[14]

$$\sigma = C_\sigma a_1^{N_1} a_2^{N_2} a_3^{N_3} \ldots a_M^{N_M} \tag{3}$$

In the general nonrelativistic case of a thermal quantum plasma, of which the classical thermal plasma is a special case, the secondary conductivity quantity σ depends on the dimensional primary quantities $a_1 = e$ (electron charge), $a_2 = m$ (electron mass), $a_3 = \hbar$ (Planck's constant), $a_4 = n$ (electron density), and $a_5 = KT$ (thermal energy). The dimensionless factor C_σ is in general a function of the dimensionless parameters p/p_i of the plasma,

$$C_\sigma = C_\sigma(Z_i, m/m_i, \ldots p/p_i, \ldots) \tag{4}$$

For example, Z_i is the ratio of the magnitudes of the ion and electron charges, m/m_i the electron-to-ion mass ratio, etc. The electrical conductivity σ and its primary quantities have the following dimensions $\mathfrak{D}$ ($\mathfrak{L}$ = dimension of length, $\mathfrak{I}$ = dimension of time, $\mathfrak{M}$ = dimension of mass):

$$\mathfrak{D}[\sigma]=\mathfrak{I}^{-1} \qquad \mathfrak{D}[n]=\mathfrak{L}^{-3}$$
$$\mathfrak{D}[e]=\mathfrak{L}^{3/2}\mathfrak{M}^{1/2}\mathfrak{I}^{-1} \qquad \mathfrak{D}[KT]=\mathfrak{M}\mathfrak{L}^{2}\mathfrak{I}^{-2}$$
$$\mathfrak{D}[m]=\mathfrak{M} \qquad \mathfrak{D}[\hbar]=\mathfrak{M}\mathfrak{L}^{2}\mathfrak{I}^{-1} \tag{5}$$

It is recognized that the secondary and primary quantities depend only on three basic dimensions which are independent, namely, $\mathfrak{L}$, $\mathfrak{I}$, and $\mathfrak{M}$. For this reason, dimensional analysis provides at most three independent equations for the determination of the powers N_i, $i=1,2,3,..M$ in Eq. (3). That is, at most three powers can be calculated while at least M-3 powers have to be determined by comparison with experiments or by physical arguments.

The fundamental equation (3) is applied below to the determination of the electrical conductivity of (nonrelativistic) nonideal, classical, and quantum plasmas in which the electrons are responsible for the electric current transport. To illustrate the results, they will be expressed in terms of the classical (γ) and quantum mechanical (Γ) interaction parameters for $Z=1$ and the order of magnitude of fundamental electron energies:

$$\gamma=E_C/E_T \qquad \Gamma=E_C/E_Q \tag{6}$$
$$E_T=KT \tag{7}$$
$$E_C=e^2n^{1/3} \tag{8}$$
$$E_Q=\hbar^2/mn^{-2/3} \tag{9}$$

E_Q is the order of magnitude of the quantum potential energy $Q=-(\hbar^2/2m)\nabla^2\rho^{1/2}/\rho^{1/2}$ of an electron in the plasma. E_C and E_T are the Coulomb interaction and thermal energies, respectively.

Classical Plasma

In a classical ($\hbar\rightarrow 0$) electron plasma, in which the thermal energy KT is negligible compared with the Coulomb interaction energy, the conductivity depends on the dimensional parameters e, m, and n. By Eq. (3),

$$\sigma=C_\sigma e^{N_1}m^{N_2}n^{N_3} \tag{10}$$

where

$$(3/2)N_1-3N_3=0,\ (1/2)N_1+N_2=0,\ -N_1=-1 \tag{11}$$

by comparison of the powers N_i of $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{I}$ in Eq. (10). These are three independent equations, which determine N_1, N_2, and N_3 uniquely,

$$N_1=1,\quad N_2=-1/2,\quad N_3=1/2 \tag{12}$$

By Eqs. (10) and (12), the conductivity of the zero-temperature, classical plasma is

$$\sigma=C_\sigma en^{1/2}/m^{1/2} \tag{13}$$

A similar result was first derived by Buneman[15] and later by Hamberger and Friedman[16] for a $T=0$ plasma with predominant space change turbulence interaction, by means of semiquantitative physical arguments, which give $C_\sigma=\pi^{1/2}(m_i/m)^{1/3}$ for $Z=1$.

In a classical ($\hbar\rightarrow 0$), thermal ($KT>0$) electron plasma, the conductivity depends on the dimensional parameters e, m, n, and KT. By Eq. (3)

$$\sigma=C_\sigma e^{N_1}m^{N_2}n^{N_3}(KT)^{N_4} \tag{14}$$

where

$$\frac{3}{2}N_1-3N_3+2N_4=0,\quad \frac{1}{2}N_1+N_2+N_4=0$$
$$-N_1-2N_4=-1 \tag{15}$$

by comparison of the powers N_i of $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{I}$ in Eq. (14). These are three independent equations, which determine three of the four powers in terms of the fourth,

$$N_1=1-2N,\ N_2=-1/2,\ N_3=1/2-1/3N,\ N_4\equiv N \tag{16}$$

Collecting of powers of N in Eq. (14), condenses the conductivity formula to

$$\sigma=C_\sigma(KT/e^2n^{1/3})^N en^{1/2}/m^{1/2} \tag{17}$$

For an ideal, classical plasma, $\sigma=(ne^2/m)\tau$ cannot depend on a power of n since $\tau\sim n^{-1}$ for binary e-i collisions. Hence, σ and N are for an ideal $T>0$ plasma

$$\sigma=C_\sigma(KT)^{3/2}/e^2m^{1/2},\quad N=3/2 \tag{18}$$

in agreement with kinetic theory,[12] which shows that $C_\sigma=3/4(2\pi)^{1/2}Z\ell n\Lambda\sim 10^{-1}$. For $N=0$, Eq. (17) reduces to σ of the classical $T=0$ plasma [Eq. (13)].

For a nonideal (classical) plasma, in which ν-body interactions dominate, we have $\tau\propto n^{1-\nu}$ and $\sigma=(ne^2/m)\tau\propto n^{2-\nu}$, with $\nu>2$, i.e., $N=3\nu-9/2>3/2$. Thus, we find for the conductivity of nonideal, classical plasmas (ν-body interactions)

$$\sigma=C_\sigma\gamma^{-N}en^{1/2}/m^{1/2},\quad N>3/2 \tag{19}$$

where $\gamma=E_C/E_T$ is the nonideality parameter defined in Eqs. (6-8). Equation (19) expresses the interesting result that the conductivity of a nonideal, classical plasma decreases proportional to γ^{-N} with increasing nonideality γ since $N>3/2$. The exact value of N can be determined by comparison with experimental data.

Quantum Plasma

In a completely degenerate electron plasma, $E_T\ll E_Q$, the conductivity depends on the dimensional parameters e, m, n, and $\hbar$, but not on KT. By Eq. (3),

$$\sigma=C_\sigma e^{N_1}m^{N_2}n^{N_3}\hbar^{N_4} \tag{20}$$

Comparison of the powers of $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{I}$ in Eq. (20) gives three independent equations for the N_i, which determine three of the four powers in terms of the fourth,

$$N_1=1-2N,\ N_2=-1/2-N,\ N_3=1/2+1/3N,\ N_4\equiv 2N \tag{21}$$

Hence, the conductivity of the completely degenerate ($T=0$) electron plasma is

$$\sigma=C_\sigma\left(\frac{\hbar^2n^{1/3}}{me^2}\right)^N en^{1/2}/m^{1/2} \tag{22}$$

For $N=3/2$, Eq. (22) leads to the conductivity of the solid metal at $T=0$,

$$\sigma=C_\sigma\hbar^3n/e^2m^2,\quad N=3/2 \tag{23}$$

where[17] $C_\sigma\propto Z^{-1}$. For $N=0$, Eq. (22) reduces to σ of the classical $T=0$ plasma [Eq. (13)].

For complete degeneracy, E_T is negligible compared with $E_Q\sim mv_F^2$, where $v_F\sim(\hbar/m)n^{1/3}$ is the Fermi velocity. For this reason, Eq. (22) is rewritten in the form

$$\sigma = C_\sigma \Gamma^{-N} e n^{1/3} / m^{1/2} \tag{24}$$

Since $N=3/2$ for the $T=0$ metal, it is to be expected that $N>0$ for the completely degenerate electron plasma, i.e., its conductivity decreases with increasing nonideality Γ [Eq. (24)]. This formula is useful for the interpretation of conductivity data of completely degenerate electron plasmas, with N as the adjustable parameter.

In a partially degenerate electron plasma, $E_T \lesssim E_Q$, the conductivity depends on the dimensional parameters e, m, n, KT, and $\hbar$. By Eq. (3),

$$\sigma = C_\sigma e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \hbar^{N_5} \tag{25}$$

Comparison of the powers of $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{J}$ in Eq. (25) gives three independent equations, which permit to express three of the five powers in terms of the remaining ones,

$$N_1 = 1 - 2A - 2B, \qquad N_2 = -\frac{1}{2} - B$$

$$N_3 = \frac{1}{2} - \frac{1}{3}A + \frac{1}{3}B, \qquad N_4 \equiv A, \quad N_5 \equiv 2B \tag{26}$$

Hence, the conductivity of the $T>0$ quantum plasma is

$$\sigma = C_\sigma \left(\frac{KT}{e^2 n^{1/3}}\right)^A \left(\frac{\hbar^2 n^{1/3}}{me^2}\right)^B e n^{1/2} / m^{1/2} \tag{27}$$

where A and B are powers which cannot be determined by dimensional reasoning.

For $A=-1$ and $B=-3/2$, Eq. (27) yields the conductivity of solid metals at temperatures $T>0$,

$$\sigma = C_\sigma \frac{me^6}{\hbar^3} \frac{n^{1/3}}{KT}, \quad A=-1, \quad B=-3/2 \tag{28}$$

where[17] $C_\sigma \propto Z^{-1/3}$. Equation (28) expresses the $1/T$ law[17] of the metallic conductivity at "high temperatures."

Equation (27) contains the dimensionless groups γ and Γ, which permit rewriting the conductivity formula as

$$\sigma = C_\sigma \gamma^{-A} \Gamma^{-B} e n^{1/2} / m^{1/2} \tag{29}$$

Since $A=-1$ for $T>0$ metals [Eq. (28)] and $B=3/2$ for $T=0$ metals [Eq. (23)], one can speculate that $A \gtrsim -1$ and $B>0$ for nonideal, partially degenerate plasmas. A and B can readily be determined by means of conductivity measurements for nonideal quantum plasmas.

The dimensionless coefficient C_σ can be determined experimentally or by calculation of the momentum exchange of the electrons from a kinetic equation for nonideal classical and quantum plasmas, respectively. For ideal and weakly nonideal plasmas, $0 \leq \gamma \ll 1$, the coefficient C_σ varies very little[12] with n and T. For proper nonideal plasmas, C_σ is a true constant when many particle interactions are dominant. For these reasons, the coefficient C_σ is of secondary importance concerning the dependence of the conductivity σ on the variables n and T.

Conclusion

We have derived new conductivity formulas for nonideal classical $T>0$ plasmas [Eq. (17) or (19)], completely degenerate plasmas [Eqs. (22) and (23)], and nonideal $T>0$ quantum plasmas [Eqs. (27) and (28)]. These formulas can be used to interprete conductivity measurements on nonideal plasmas. Once the still undetermined powers N, A, and B are known empirically, it should be possible to develop a conductivity theory for nonideal plasmas that provides an explanation of the γ and Γ dependence from microscopic principles.

Acknowledgment

This work was supported by the U. S. Office of Naval Research.

References

[1] Franck, E. U., and Hensel, F., "Metallic Conductance of Supercritical Mercury Gas at High Pressure," *Physical Review*, Vol. 147, 1966, pp. 109-110.

[2] Renkert, H., Hensel, F., and Franck, E. U., "Metal-Nonmetal Transition in Dense Cesium Vapor," *Physics Letters*, Vol. 30A, 1969, pp. 494-495.

[3] Barolskii, S. H., Ermokhin, N. V., Kulik, P. P., and Melnikov, V. M., "Measurement of Electric Conductivity of Dense Strongly Nonideal Cesium Plasma," *Soviet Physics, Journal of Experimental and Theoretical Physics*, Vol. 35, 1976, pp. 94-97.

[4] Ivanov, Yu. V., Mintsev, V. B., Fortov, V. E., and Dremin, A. N., "Electric Conductivity of Nonideal Plasma," *Soviet Physics, Journal of Experimental and Theoretical Physics*, Vol. 44, 1976, pp. 112-116.

[5] Goldbach, C., Nollez, G., Popovic, S., and Popovic, M., "Electric Conductivity of High Pressure Ionized Argon," *Zeitschrift fuer Naturforschung*, Vol. 33a, 1977, pp. 11-16.

[6] Ermokhin, N. V., Kovalec, B. M., Kulik, P. P., and Ryabyi, V. A., "Temperature Dependence of Electric Conduction of Dense Cesium Plasma," *Journal de Physique*, Vol. 39, Supp. 5, 1978, pp. 200-204.

[7] Wilhelm, H. E., "Momentum and Energy Exchange Between Beams of Particles Interacting by Yukawa Type Potentials," *Physical Review*, Vol. 187, 1969, pp. 382-392.

[8] Alekseev, V. A., "Measurement of Electric Conductivity and Density of Metals in the Supercritical State," *High Temperature*, Vol. 8, 1970, pp. 597-611.

[9] Ermokhin, N. V., Kovalev, B. M., Kulik, P. P., and Ryabyi, V. A., "An Experimental Investigation of Coulomb Interactions in Dense Plasmas," *High Temperature*, Vol. 9, 1971, pp. 611-618.

[10] Lomakin, B. N. and Fortov, V. E., "Equation of State of Nonideal Cesium Plasma," *Soviet Physics, Journal of Theoretical and Experimental Physics*, Vol. 36, 1973, pp. 48-53.

[11] Klimontovich, Y. L., "Kinetic Equations for a Classical Nonideal Plasma," *Soviet Physics, Journal of Experimental and Theoretical Physics*, Vol. 35, 1972, pp. 920-925.

[12] Spitzer, L., *Physics of Fully Ionized Gases*, Interscience Publishers, Inc., New York, 1956, pp. 77-83.

[13] Dupre, J., *Theorie Mechanique de la Chaleur*, Masson & Cie, Paris, 1869, pp. 1-60.

[14] Buckingham, E., "Similar Systems and Dimensional Equations," *Physical Review*, Vol. 4, 1914, pp. 345-376.

[15] Buneman, O., "Dissipation of Currents in Ionized Media," *Physical Review*, Vol. 115, 1959, pp. 503-517.

[16] Hamberger, S. M., and Friedman, M., "Electrical Conductivity of Highly Turbulent Plasma," *Physical Review Letters*, Vol. 21, 1968, pp. 674-677.

[17] Sommerfeld, A., and Bethe, H., "Electron Theory of Metals," *Handbook of Physics*, Vol. 24/2, Springer Verlag, Berlin, 1933, pp. 499-536.

AIAA 81-4095

Longitudinal Grooves for Bluff Body Drag Reduction

B. Quass,* F. Howard,† L. Weinstein,* and D. Bushnell‡

NASA Langley Research Center, Hampton, Va.

Nomenclature

C_D = aerodynamic drag coefficient based on maximum cross-sectional area

D = body diameter, 6.081 cm

R = planform fairing radius at cylinder-cone junction

$R_{\infty,D}$ = freestream Reynolds number based on body diameter

Received July 11, 1980. This paper is declared a work of the U.S. Government and therefore is in the public domain.

*Aerospace Engineer, Fluid Mechanics Branch, High-Speed Aerodynamics Division. Member AIAA.

†Aerospace Engineer, Fluid Mechanics Branch, High-Speed Aerodynamics Division.

‡Head, Fluid Mechanics Branch, High-Speed Aerodynamics Division. Associate Fellow AIAA.